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Influence of the cavitation on the stress–strain fields of compressible Blatz–Ko materials at finite deformation

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Abstract

The aim of this article is the analysis of fracture growth in media characterized by random distribution of micro-failure mechanisms per unit volume. The deformation behavior of the material was investigated in terms of a spherical unit cell model, containing an initially spherical cell of porous. The *effective* elastic bulk modulus as a function of micro-failures concentration was computed and using the Griffith critirium and certain boundary conditions the rate at which the void area varies was determined too. Along the analysis a special form of the strain energy function for compressible Blatz–Ko material was used. The applied traction on the unit cell of the material was determined as a function of the porosity of the material, as well as the strain field within the solid. At low values of the porosity, as the applied external traction was increased instabilities were observed in the void growth.

Keywords: Porosity; Effective moduli; Griffith criterion; Blatz–Ko materials; Finite deformation; Failure mechanisms

1. Introduction

Materials in the applications come in a variety of forms with a very wide range of properties. Material science is utilized to gain a fundamental understanding of materials behavior in heterogeneous systems, so that further improvements can be achieved. Only a rigorous discipline of heterogeneous materials behavior can provide the key of optimizing material utilization. *Stiffness* and *strength* are by far the most important of all the engineering properties. The mathematical characterization of the stiffness property is the basis of the methods of analysis that ultimately lead to design using engineering materials. In the literature different types of heterogeneous media are considered with primary emphasis on the cases of spherical, cylindrical and lamellar geometric forms of inclusion embedded in a continuous matrix phase. For heterogeneous materials focus is concentrated on determining the effective stiffness properties, i.e. the average measures of

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the stiffness of the material, taking into the account the properties of all phases and their interaction. In some cases is possible to obtain exact solutions for the effective moduli, when this is not possible then it is possible to obtain *bounds* on the effective moduli.

For two-material heterogeneous system, one material being continuous and the other in the form of discrete inclusions, Russel and Acrivos (1972) have found the effective properties of this system. It was shown that only the conditions within the inclusions are needed for the evaluation of the effective properties tensor, C_{ijkl}, for non-dilute conditions. Eshelby (1957) has proved that an ellipsoidal inclusion, embedded in an infinite medium, is in the state of homogeneous deformation, corresponding to that imposed in an infinite media, at large distances from the inclusion. For a dilute elastic suspension of spherical particles Eshelby (1957) was able to derive simple formulae for the shear and bulk effective moduli. Hashin (1962) introduced the *composite spheres model* (CSM) that is composed of a gradation of sizes of spherical particles embedded in a continuous matrix phase. The size distribution is not random, but rather has a very particular characteristic. Following Hashin's (1962) approach one can determine the lower and upper bounds for the effective moduli of non-dilute suspensions. Another approach, of determining effective moduli for heterogeneous media, was based on the *three-phase model* proposed by Kerner (1956) and Van der Pol (1958).

Richard (1975) has given experimental data of the effective uniaxial modulus E, for a suspension of spherical particles. The composite material contained a suspension of glass micro-spheres embedded in a polyester matrix material. Christensen and Lo (1979) have found the solutions for effective shear properties in three phase sphere and cylinder models. Hershey (1954) proposed the *self-consistent scheme* to model the behavior of polycrystalline materials. Hill (1965) and Budiansky (1965) extended the *self-consistent scheme* to multiphase materials. According to Budiansky (1965) approach each phase of the composite is viewed as being lumped as a single ellipsoid inclusion in an infinite matrix of the unknown effective properties of the problem. Applying uniform stress or strain conditions at infinity then allows the determination of the average conditions in the inclusion. After this operation has performed for all phases, the average conditions are known for all phases, in terms of the individual phase properties and the effective properties. Hence, average conditions in the entire composites are known and the effective moduli can be calculated from the averages (see Christensen, 1979).

Kakavas and Chang (1991, 1992) have applied the well-known acoustic emission experimental technique, to study the void production in rubber-like materials subjected to triaxial stress conditions. Blatz and Kakavas (1993) have given theoretical insight, for the porosity of such materials. Several researches have performed theoretical studies on the influence of void growth in polymers. Haward and Owen (1973) have performed elastic–plastic modeling to the analysis of a planar model with cylindrical voids. Huang and Kinloch (1992) considered an axisymmetric model with spherical voids and Steenbrink et al. (1997) has also studied the voids in amorphous glassy polymers that exhibit elastic–viscoplasticity with rate dependent yield, intrinsic softening and progressive strain hardening at large strains. They also examined the plastic deformation around initially spherical voids, using axisymmetric cell analysis, and their resulting growth in terms of size and shape to large strains. A constitutive model proposed by Steenbrink et al. (1997) for the description of the macroscopic overall behavior of *porous* glassy polymers.

In this article, the effect of void volume fraction on the strain and stress field in compressible media is presented. The material is assumed geometrically non-linear and is characterized from a special form of strain energy function, the so-called Blatz–Ko (1962). Basic theoretical background of the kinetics of finite deformation of non-linear elasticity is depicted in Section 2, while the formulation of the problem in spherical coordinates is shown in Section 3. A model for the computation of the effective modulus of porous media was developed in Section 4 of this article, and the dependence of the stress and strain fields on the porosity of the material is also shown in the same section. The derived numerical results based on the proposed theory are shown in Section 5 of this study.

2. Kinematics of finite deformation

In this section, some basic theoretical background is presented for compressible materials subjected to large deformation. Following Beaty (1987) and/or Ogden (1984) notation, a body $\mathbf{B} = \{P_k\}$ is a set of material points P_k that are called particles. The motion of the particle P relative to a reference frame $\{O, e_i\}$ is described by the time *locus* of its position vector $\mathbf{x}(P,t)$ relative to $\{O, e_i\}$, where this locus is the trajectory or path of P in $\{O, e_i\}$. Its position vector $\mathbf{X}(P)$ in $\{O, e_i\}$ may identify a particle P, at some reference time. The domain $\boldsymbol{\kappa}_R$ of $\mathbf{X}$ is called a reference configuration of $\mathbf{B}$. The motion of a typical particle P from $\boldsymbol{\kappa}_R$, is described by the vector function:

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) \Longleftrightarrow x_i = \chi_i(X_\alpha, t) \quad \forall i, \alpha = 1, 2, 3 \tag{1}$$

The domain $\boldsymbol{\kappa}$ of $\mathbf{x}$, the region occupied from the current configuration of $\mathbf{B}$, at time t, is called the *current configuration* of $\mathbf{B}$. As is shown in Beaty (1987) review article (see Fig. 1), $\mathbf{x}$ denotes the place at time t in the current configuration $\boldsymbol{\kappa}$ which is occupied by the particle P whose place was $\mathbf{X}$ in the reference configuration $\boldsymbol{\kappa}_R$. In particular, if one compares the two configurations without requiring knowledge of the intermediate stages in motion, so that the time dependence in Eq. (1) is not needed, then Eq. (1) in components form can be rewritten as

$$x_i = \chi_i(X_\alpha) \quad \forall i, \alpha = 1, 2, 3 \tag{2}$$

Taken the variation of Eq. (2) one can write that:

$$\mathrm{d}x_i = F_{ij}\,\mathrm{d}X_j \quad \forall i, j = 1, 2, 3 \tag{3}$$

where F_{ij} is a second order tensor defined by

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \tag{4}$$

The tensor F_{ij} is called the *deformation gradient* tensor relative to the undeformed state B_0. This tensor transforms the element $\mathrm{d}\mathbf{X}$ of a material line in $\boldsymbol{\kappa}_R$ into the element $\mathrm{d}\mathbf{x}$ of the deformed image line in $\boldsymbol{\kappa}$. With respect to the deformation gradient tensor F_{ij}, one can write the *right* and *left* Cauchy–Green deformation tensors as follows (Truesdell and Null, 1965)

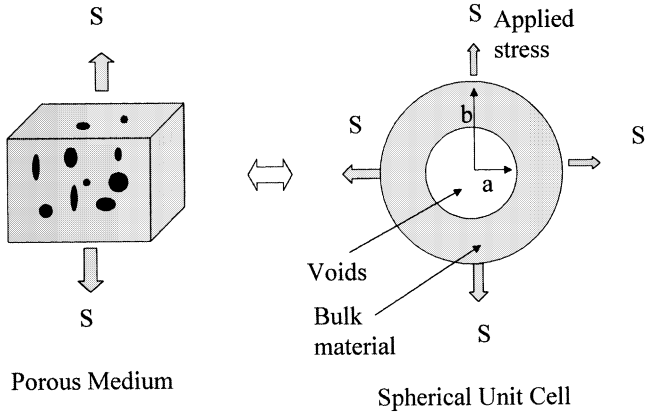

Fig. 1. A representation of a porous medium with a spherical unit cell described by a Blatz–Ko strain energy function. The sphere of the bulk material has a radius b, and that of the porous medium is a.

$$\mathbf{C} = \mathbf{F}^{\mathbf{T}}\mathbf{F},\ \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathbf{T}} \Longleftrightarrow C_{ij} = F_{ki}F_{kj},\ B_{ij} = F_{ik}F_{jk} \tag{5}$$

where the tensors $\mathbf{C}$ and $\mathbf{B}$ are positive and symmetric.

Letting the volume of an infinitesimal element of the material in the undeformed and deformed configuration being $\mathrm{d}V$ and $\mathrm{d}v$ respectively, then one can define the Jacobian of the system as follows:

$$J \equiv \det(\mathbf{F}) = \frac{\mathrm{d}v}{\mathrm{d}V} \tag{6a}$$

By *convention* one can define volume elements to be positive so that relative orientation of the line elements is preserved under deformation. It follows from Eq. (6a) that:

$$J \equiv \det \mathbf{F} \geqslant 0 \tag{6b}$$

In components J represents the Jacobian determinant $\det(\partial x_i/\partial X_\alpha)$ of the deformation (2). Physically, the variable J, represents the local ratio of current to reference volume of a material volume element. In a material where the volume does not change Eq. (6a) is rewritten as

$$J \equiv \det(\mathbf{F}) = 1 \tag{6c}$$

and the deformation is called *isochoric*.

The measure of the energy stored in a material as a result of the deformation is called *elastic stored energy function* and is denoted by W. For an isotropic hyperelastic material holds:

$$W(\mathbf{C}) = W(\mathbf{B}) \tag{7a}$$

This relation shows that for a given deformation the strain energy function has the same values whether $\mathbf{C}$ or $\mathbf{B}$ is used as the independent variable. The principal values of the tensors $\mathbf{C}$ and $\mathbf{B}$ are the same, hence the principal invariants $I_i(\mathbf{C})$ $(i = 1, 2, 3)$ and $I_i(\mathbf{B})$ are the same for every deformation. Therefore Eqs. (7a) and (7b) suggests that the strain energy must be *isotropic* scalar valued function of these principal invariants, i.e.

$$W = W(I_1, I_2, I_3) \tag{7b}$$

where the invariants I_i, are given by (see Bonet and Wood, 2000)

$$I_1 = \mathrm{tr}(\mathbf{B}), \quad I_2 = (I_1^2 - \mathrm{tr}(\mathbf{B}^2))/2, \quad I_3 = \det(\mathbf{B}) \tag{7c}$$

The constitutive equation for an *isotropic hyperelastic* material may be written as

$$\mathbf{T} = \frac{2}{J}\frac{\partial W}{\partial \mathbf{B}}\mathbf{B} \tag{8}$$

where the *left Cauchy–Green deformation tensor*, $\mathbf{B}$, is provided by Eq. (5), and $\mathbf{T}$ defines the Cauchy or *true* stress tensor. Using Eq. (7b) in the last equation reveals the following form of the constitutive equation for an isotropic hyperelastic material (Beaty, 1987):

$$\mathbf{T} = \alpha_0\mathbf{1} + \alpha_1\mathbf{B} + \alpha_2\mathbf{B}^2 \tag{9a}$$

or by using the Caley–Hamilton theorem Eq. (9a) is written as

$$\mathbf{T} = \beta_0\mathbf{1} + \beta_1\mathbf{B} + \beta_{-1}\mathbf{B}^{-1} \tag{9b}$$

where the coefficients $\alpha_k (k = 1, 2, 3)$ and $\beta_m (m = 0, 1, -1)$ are called *material response functions* and they are functions of the strain invariants I_i.

Blatz and Ko (1962) have replaced the I_i invariants by J_k defined by

$$J_1 \equiv I_1 = \mathrm{tr}(\mathbf{B}), \quad J_2 \equiv \frac{I_2}{I_3} = \mathrm{tr}(\mathbf{B}^{-1}), \quad J_3 \equiv I_3^{1/2} = \det(\mathbf{F}) \tag{10}$$

Hence, the strain energy function may be rewritten as a function of J_k, i.e.

$$W = W(J_1, J_2, J_3) \tag{11}$$

Blatz and Ko (1962) and independently Beaty (1987) have proved that the *true* stress $\mathbf{T}$, can be written as

$$\mathbf{T} = W_3(J_3)\mathbf{1} + \frac{Gf}{J_3}\mathbf{B} - \frac{G(1-f)}{J_3}\mathbf{B}^{-1} \tag{12}$$

where $W_3 = \partial W/\partial J_3$, G defines the shear modulus of the material, f is a constant, and $\mathbf{1}$ defines the unit tensor. A material described by the constitutive equation (12) is called a *Blatz–Ko material*, and is the one whose elastic response functions depend on J_3 alone.

In terms of the principal stretches λ_i $(i = 1, 2, 3)$, for *compressible* materials, Blatz and Ko (1962) have proposed the following form of W,

$$W(\lambda_1, \lambda_2, \lambda_3, J) = \frac{G}{2}\left(\sum_{i=1}^{3}\lambda_i^2 - 3\right) + Gf(J) \tag{13}$$

where, G, defines the *shear modulus* of the material, J is the volume ratio ($J = \prod_{i=1}^{3}\lambda_i$) and the properties of the function $f(J)$ will be discussed later along this analysis.

More general forms of W, for compressible materials, have also been proposed by various other researches, (see Ogden, 1984). The *engineering stress* σ_i, i.e. the applied load per unit undeformed area, is given by

$$\sigma_i = \frac{\mathrm{d}W}{\mathrm{d}\lambda_i} \quad (\forall i = 1, 2, 3) \tag{14}$$

and the corresponding components of the *true stress* t_i, i.e. the applied load per unit deformed area, is given by

$$t_i = \frac{\lambda_i}{J}\frac{\mathrm{d}W}{\mathrm{d}\lambda_i}(\text{no summation on } i) \tag{15}$$

In terms of the principal Cauchy (*true*) stresses, t_i, the *equilibrium* equation, in spherical coordinates, and in radial direction is given by

$$r\frac{\mathrm{d}\hat{t}_r}{\mathrm{d}r} + 2(\hat{t}_r - \hat{t}_\vartheta) = 0 \tag{16}$$

where $\hat{t}_r = t_r/G$, $\hat{t}_\vartheta = t_\vartheta/G$ are the normalized *radial* and *tangential* components of the *true* stress, respectively.

3. Problem formulation

A spherical unit cell model (see Christensen, 1979) is used in order to investigate the mechanical behavior of porous materials. One can consider a sphere of a compressible material having an initial radius b_0 and containing a spherical void of radius a_0, which is subjected to a uniform radial true (Cauchy) traction, S, on its outer boundary (see Fig. 1). When the sphere is made from a compressible Blatz–Ko material, characterized by an initial shear modulus G, the principal Cauchy (*true*) stress components t_α $(\alpha = 1, 2, 3)$ are related to the principal stretches λ_α, by the following equation:

$$\hat{t}_\alpha = \frac{1}{J}\lambda_a^2 + f'(J) \quad (\forall \alpha = 1, 2, 3) \tag{17}$$

where $f'(J) = \mathrm{d}f/\mathrm{d}J$.

The sphere, which is consisted from the porous material, is considered to have zero stiffness. The initial size of the *void* cell is denoted by a_0, and that of the *bulk* material by b_0, while the deformed size of porous and material cell is a, and b respectively. The bulk material will dilate very slightly, while the porous material will dilate more or less highly depending upon the applied tension. Due to spherical symmetry of the problem, the deformed coordinate in the *radial* direction, is related to the undeformed coordinate, by the following equation:

$$r = r(R) \tag{18a}$$

and the *principal stretch* corresponding to *radial* direction is

$$\lambda_r = \frac{\mathrm{d}r}{\mathrm{d}R} \tag{18b}$$

while the principal stretches in φ and ϑ directions are (Ogden, 1984)

$$\lambda_\varphi = \lambda_\vartheta = \frac{r}{R} \equiv \lambda \tag{19}$$

The volume ratio, J, for a *compressible* material is

$$J = \prod_{i=1}^{3} \lambda_i = \lambda_r \lambda^2 \Rightarrow \lambda_r = \frac{J}{\lambda^2} \tag{20}$$

where the ratio of the change in volume $J = \mathrm{d}v/\mathrm{d}V$, is considered constant.

The last equation expresses the radial component of the principal stretch λ_r, as a function of the tangential λ.

4. Effective bulk modulus of porous compressible hyperelastic media

Substitution of Eq. (13) into Eq. (14) the *engineering stress,* can be computed as follows:

$$\frac{\sigma_i}{G} = \frac{J}{\lambda_i}\frac{t_i}{G} = \lambda_i + \frac{J}{\lambda_i} f'(J) \quad \forall i = 1,2,3 \tag{21}$$

In *spherical* coordinates Eq. (21) can be written as

$$\begin{aligned} \frac{\sigma_r}{G} &= \frac{J}{\lambda^2} + \lambda^2 f'(J) \\ \frac{\sigma_\varphi}{G} &= \frac{\sigma_\vartheta}{G} = \lambda + \frac{J}{\lambda} f'(J) \end{aligned} \tag{22}$$

where $\lambda = 1 + \varepsilon$ (ε being the applied strain in radial direction).

For *small* strains Eq. (22) follows Hooke's law, and hence the function f, for $J = 1$, must satisfies the limited conditions:

$$f(1) = 0, \quad \frac{\mathrm{d}f}{\mathrm{d}J}(J=1) = -1, \quad \frac{\mathrm{d}^2 f}{\mathrm{d}J^2}(J=1) = \frac{1}{1-2v} = \frac{K}{G} + \frac{1}{3} \equiv A \tag{23}$$

where the constants K and G represent the *bulk* and *shear* modulus of the material, respectively and A is a constant which will be determined later on.

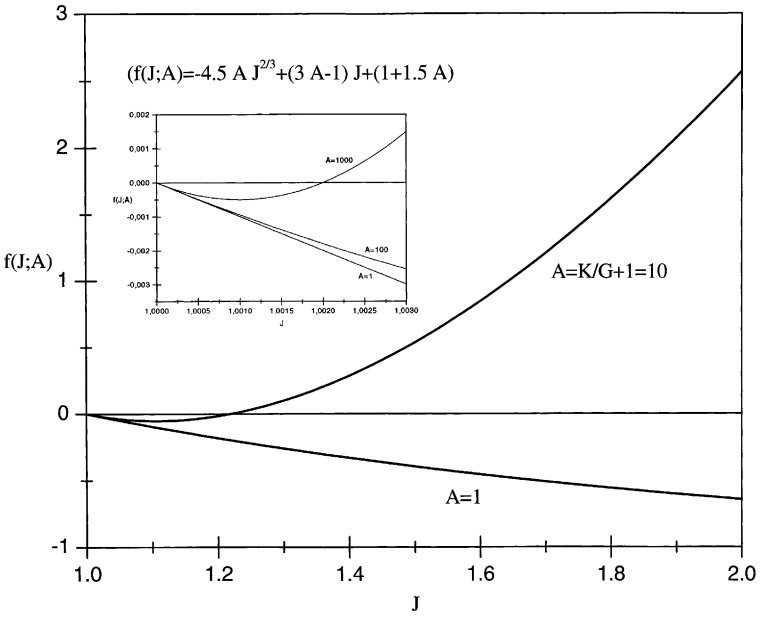

Fig. 2. Representation of the function $f(J)$ vs. the Jacobian of the system J, for various values of $A = K/G + 1/3$.

One obvious function which satisfies the limited restrictions on f, is given by the following expression:

$$f(J) = -\frac{9}{2}AJ^{2/3} + (3A-1)J + \left(1 + \frac{3}{2}A\right) \tag{24}$$

where A is defined in Eq. (23) and obviously one can recognize that the function $f(J)$ is *non-linear*. A plot of the function $f(J)$ for two different values of A (e.g. $A = 1$, 10) is shown in Fig. 2. In addition in the insertion one can see the behavior of the function $f(J)$ for large values of A, e.g. $A = 100$, 1000. In case of $A = 1000$ the bulk modulus is much larger that the shear modulus of the material, i.e. the solid is incompressible.

Using Eqs. (18a,18b) and (19) all the derivatives on the λ, can be transferred on the *deformed* coordinate r, i.e.

$$r\frac{\mathrm{d}}{\mathrm{d}r} = \left(\frac{J}{\lambda^2} - \lambda\right)\frac{\mathrm{d}}{\mathrm{d}\lambda} \tag{25}$$

where J is considered to be a constant.

Replacing Eq. (25) into Eq. (16) yields:

$$\left(\frac{J}{\lambda^2} - \lambda\right)\frac{\mathrm{d}\hat{t}_r}{\mathrm{d}\lambda} + 2(\hat{t}_r - \hat{t}_\vartheta) = 0 \tag{26}$$

Substitution of the components of the *radial* and *tangential* stresses from Eq. (22), and using Eq. (19), then Eq. (21) yields the following non-linear first order differential equation:

$$\frac{\mathrm{d}J}{\mathrm{d}\lambda} = F(J,\lambda) \equiv -\frac{2(\lambda^3 - J)}{\lambda(1 + A\lambda^4 J^{-4/3})} \tag{27}$$

Letting $J = (\omega\lambda)^3$, where ω is a function of λ, then Eq. (27) is transferred to the following separable form:

$$\frac{\mathrm{d}\lambda}{\lambda} = \mathrm{d}\ln(\lambda) = G(\omega)\mathrm{d}\omega \equiv -\frac{(A+\omega^4)\mathrm{d}\omega}{\omega(A+\frac{2}{3}\omega+\frac{1}{3}\omega^4)} \tag{28}$$

On the *outer* surface of the spherical shell (i.e. at $r = b$), since in the linear region the deformation is small, the variables λ, J, ω have the limited values $\lambda \to 1+\varepsilon, J \to 1+\mathrm{O}(\varepsilon^2), \omega \to 1-\varepsilon$. In the *inner* surface of the spherical cell (i.e. $r = a$), it can show that $\lambda \to 1/\varepsilon, J \to 1+\mathrm{O}(\varepsilon^2), \omega \to \varepsilon$. Hence, the variables λ and ω are related through the equation

$$\mathrm{d}\ln(\lambda) \approx -\mathrm{d}\ln(\omega) \tag{29a}$$

or equivalently

$$\lambda\omega = B \tag{29b}$$

where B is an undetermined constant, which must be, determined by the boundary conditions of the problem. Eq. (29a) implies that:

$$J = B^3 = \text{constant} \tag{29c}$$

Substitution of Eq. (24) into Eqs. (13) and (15) yields the relation between the principal stretch λ, and the deformed coordinate, r, i.e.

$$\lambda(r) = \sqrt[3]{B^3 + \Gamma^3/r^3} \tag{30}$$

where Γ is a new undetermined constant, which must be determined from the boundary conditions of the problem.

The first of Eq. (22) can be rewritten in terms of the *true* stress, as follows:

$$\hat{t}_r = \lambda_r\hat{\sigma}_r = \frac{J}{\lambda^4} + Jf'(J) \tag{31a}$$

where $\hat{t}_r \equiv t_r/G$ and $\hat{\sigma}_r \equiv \sigma_r/G$.

Substitution of the derivative of $f(J)$ from Eq. (24), into Eq. (31a) yields:

$$\hat{t}_r = \frac{J}{\lambda^4} + 3A(1-J^{-1/3}) - 1 \tag{31b}$$

where, A, was defined in Eq. (23). At the free state of stress, the Jacobian is equal to unit, and the principal stretch λ is equal to unit too, hence Eq. (31b) yields zero radial true stress. Substitution the values of J and λ from Eqs. (29c) and (30) into Eq. (31b) yields:

$$\hat{t}_r = B^3\left(B^3 + \frac{\Gamma^3}{r^3}\right)^{-4/3} + 3A(1-J^{-1/3}) - 1 \tag{32}$$

On the *outer* surface of the spherical cell the true radial stress is equal to the applied stress, S, and in the inner surface vanishes, i.e.

$$\begin{aligned} \hat{t}_r &= \hat{S} \quad \text{at} \quad R = b \\ \hat{t}_r &= 0 \quad \text{at} \quad R = a \end{aligned} \tag{33}$$

where $\hat{S} \equiv S/G$.

In order to gain insight how the hole grows, an exact geometrically non-linear analysis for an *incompressible* material (i.e. $J = 1$) is presented. Replacing the boundary conditions (33) into Eq. (32) yields:

$$\hat{S} = B^3\left(B^3 + \frac{\Gamma^3}{b^3}\right)^{-4/3} + 3A(1 - J^{-1/3}) - 1 \tag{34a}$$

$$0 = B^3\left(B^3 + \frac{\Gamma^3}{a^3}\right)^{-4/3} + 3A(1 - J^{-1/3}) - 1 \tag{34b}$$

where A, B, Γ are constants.

For *compressible* materials, in spherical coordinates, it may be easily proved that the principal stretches λ_a and λ_b, of the inner the outer surface, are related by the following relation: (Ogden, 1984)

$$\lambda_a^3 = \lambda_b^3 - \left(\frac{1 - \alpha_p}{\alpha_p}\right)J \tag{35a}$$

where α_p is the volume fraction of voids ($\alpha_p = a_0/b_0$), and the *principal* stretches λ_a, λ_b are given by

$$\lambda_a = \frac{a}{a_0}, \quad \lambda_b = \frac{b}{b_0} \tag{35b}$$

Using Eqs. (28) and (34a,34b), the *effective* bulk modulus of the material as a function of the porosity is given by

$$\frac{K_{\text{eff}}}{G} = \frac{1 - \alpha_p}{1 + \frac{3}{4}\frac{K}{G}\alpha_p} \tag{36}$$

where K, G are the *bulk* and *shear* modulus of the material. Eq. (36) is identical to the upper bound proposed by Hashin and Shtrikman (1963).

5. Griffith criterion and stress–strain fields in porous media

Griffith (1921) in his famous paper considered a two dimensional problem for a plane with a single linear crack and derived a criterion for its propagation in a linearly elastic and hence brittle material. In this study, following Griffith's approach, a relation between the applied stress on the spherical cell and the porosity of the material can be written as follows:

$$S = (1 - \alpha_p)\left[\frac{8}{9}\frac{E\gamma_s}{b(1 - v)}\right]^{1/2} \tag{37}$$

where E, v define the modulus of elasticity, and Poisson ratio of the bulk material, respectively, γ_s denotes the *specific surface energy* and b is the *outer* radius of the spherical cell.

After some tedious mathematical manipulations, in the limit of *incompressible* materials, Eqs. (34a) and (34b) yields:

$$\frac{S}{G} = \frac{1}{2}\left\{\frac{1}{\lambda_b^4} - \left(1 + \frac{\lambda_b^3 - 1}{\alpha_p}\right)^{-4/3} + 2\left[\frac{1}{\lambda_b} - \left(1 + \frac{\lambda_b^3 - 1}{\alpha_p}\right)^{-1/3}\right]\right\} \tag{38a}$$

$$\frac{4\gamma_s}{Gb} = \alpha_p^{1/3}\left[2\left(1 + \frac{\lambda_b^3 - 1}{\alpha_p}\right)^{2/3} + \left(1 + \frac{\lambda_b^3 - 1}{\alpha_p}\right)^{-4/3} - 3\right] \tag{38b}$$

where for small strains hold: $\lambda_b^3 - 1 \approx 3\varepsilon_b$.

For *small strain* i.e. $(3\varepsilon_b/\alpha_p) \to 0$, Eq. (38b) leads to the dependence of strain on the porosity, i.e.

$$\varepsilon_b = \left(\frac{\gamma_s}{3Gb}\right)^{1/2} \alpha_p^{5/6} \tag{39}$$

For *large strain* i.e. $(3\varepsilon_b/\alpha_p) \to \infty$, Eq. (38b) yields:

$$\varepsilon_b = \left(\frac{4\gamma_s}{Gb}\right)^{3/2} \alpha_p^{1/2} \tag{40}$$

Eqs. (39) and (40) lead to the conclusion that for *large strain* the porosity increases quadratically while for *small strains* decreases according to (6/5)th power.

For small and large values of strain the stress field as a function of the porosity is given by

For *small strain*:

$$\frac{S}{G} = \frac{4\sqrt{3}}{3}\left(\frac{4\gamma_s}{Gb}\right)^{1/2} \frac{1-\alpha_p}{\alpha_p^{1/6}} \tag{41a}$$

while for *large strains*:

$$\frac{S}{G} = 2.5 - 2\left(\frac{Gb}{2\gamma_s}\right)^{3/2} \alpha_p^{1/2} \tag{41b}$$

where b is the radius of the deformed sphere.

6. Numerical results and discussion

The variation of the strain distribution ε_b, as a function of the porosity $\alpha_p = a_0/b_0$, within the deformed material for various values of the ratio $\gamma_s^* = 4\gamma_s/Gb$ is shown in Fig. 3. The dependence of strain field on the porosity of the medium is described by Eq. (38b). The vertical axis in Fig. 3 is logarithmic and for volume fraction of voids, approximately five percent (5%), the strain field levels off. For small strains, the porosity is proportional to 1.2th power of strain (i.e. $\varepsilon_b^{1.2}$), while for large strains it is proportional to square power of strain (i.e. ε_b^2). As it can be seen from Fig. 3 the strain field is taken its higher value for porosity equal to unity.

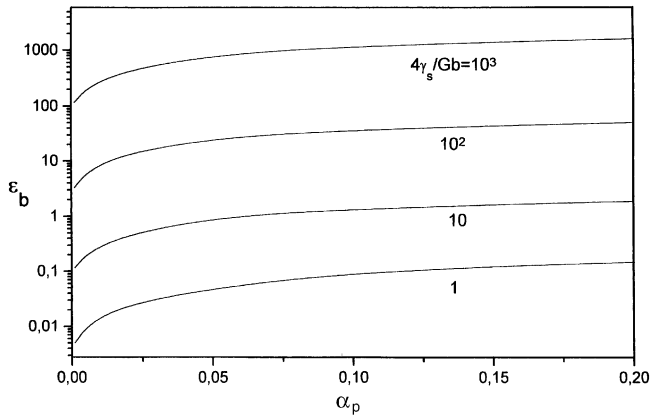

Fig. 3. Dependence of the strain field on the porosity of a porous medium for various values of the parameter $\gamma_s^* = 4\gamma_s/Gb$.

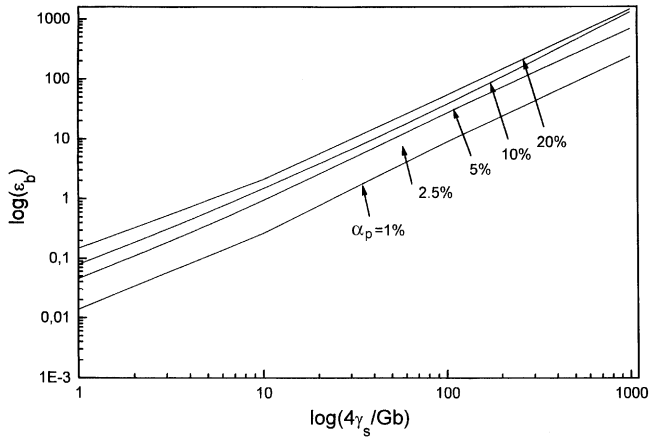

Fig. 4. Plot of $\log(\varepsilon_b)$ as a function of $\log(\gamma_s^*)$ for low values of the porosity $\alpha_p = a_0/b_0$.

A plot of $\log(\varepsilon_b)$ as a function of $\log(4\gamma_s/Gb)$ for various values of the porosity α_p, is shown in Fig. 4. From practical point of view, low values of the porosity have been taken into account. It was observed that the strain field increases drastically as the ratio $\log(4\gamma_s/Gb)$ increases too. The normalized externally applied radial true traction $\hat{S}$, on a hollow sphere consisted from an incompressible material, for various values of initial void size, f_p, as a function of the principal stretch λ_b, and is shown in Fig. 5. Similar results have also observed by Steenbrink and Van der Giessen (1999) for the *radial* expansion of a hollow *incompressible* neo-Hookean sphere with relative initial void sizes. Fig. 5 shows that up to certain stretch, λ_b^*, depending on the ratio α_p, the resistance again *dilatation* is very high until a maximum in the macroscopic stress attained. The stress beyond this point decreases monotonically as the cavity grows. Fig. 5 shows that

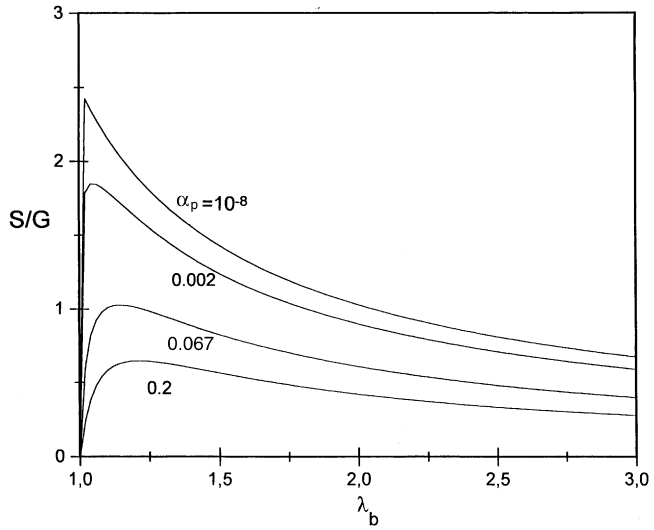

Fig. 5. Normalized externally applied stress on a hollow compressible sphere, as a function of the principal stretch for various values of the porosity.

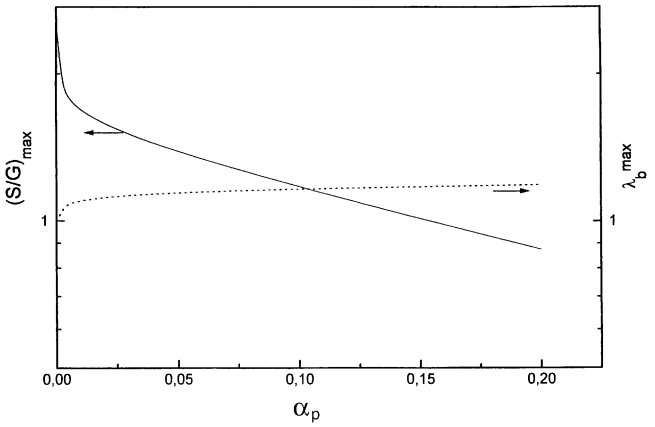

Fig. 6. Dependence of the maximum normalized applied stress and the corresponding value of the principal stretch as a function of the porosity of the material.

the limit normalized stress for the cavitation in Blatz–Ko materials leads to Gent's results (Gent and Tompkins, 1969) that is (S/G) is equal to 2.5. This is also true from Eq. (38a) as $\alpha_p \to 0$.

The maximum value of the normalized applied traction on the outer surface of the spherical cell ($r = b$) as a function of the volume fraction of voids is shown in Fig. 6. As the volume fraction of voids increases, a reduction of $\hat{S}_{max}$ is observed. At maximum point cavitation *instability* occurs and the dependence of the principal stretch at maximum stress as a function of the porosity is shown in Fig. 6 too. As the porosity of the material increases the principal stretch at maximum stress increases.

7. Conclusions

This study shows that as the strain increases in a porous medium the porosity increases too, and as the ratio $\gamma_s^* = 4\gamma_s/Gb$ takes higher values the strain field increases rapidly for fix values of the porosity. The logarithmic values of strain increase almost linearly as a function of the logarithmic values of γ_s^*. The applied stress in a porous material passes through a maximum value and after that drops monotonically as the principal stretch increases. The maximum value of the applied stress reduces and the peak is moving toward higher values as the porosity of the material increases.

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